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A NOTE ON REALIZING POLYNOMIAL ALGEBRAS

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ABSTRACT

We characterize the polynomial algebras over $\mathbf{Z}$ which are realizable as the integral cohomology of some space, under the assumption that there are not two generators in the same dimension.

Let $A = \mathbf{Z}[x_1, \cdots, x_r]$ be a graded polynomial algebra on generators x_i, $i = 1, \cdots, r$, of dimensions $\dim x_i = n_i$, $n_1 \leqq \cdots \leqq n_r$, $i = 1, \cdots, r$. A classical problem in Algebraic Topology is the following:

PROBLEM. For what values of $n_1, \cdots, n_r$ does there exist a space X such that $H^*(X; \mathbf{Z}) \cong A$?

If such a space exists we say that A is realizable. We refer to $[n_1, \cdots, n_r]$ as the type of A and if A is realizable we say that the type $[n_1, \cdots, n_r]$ is realizable. Thus the problem can be restated as follows: what types are realizable? A lot of work has been done in trying to solve this problem but it remains still open. The only known examples of realizable polynomial algebras over the integers are those isomorphic to the cohomology of products of $\mathrm{BU}(n)$, $\mathrm{BSU}(m)$, $\mathrm{BSp}(k)$, $n, m, k = 1, 2, \cdots$. The standing conjecture is that these are the only ones. In this note we affirm this conjecture and we solve the above problem in the case in which all generators have distinct degrees. We prove:

THEOREM 1. *The only types* $[n_1, \cdots, n_r]$, $n_1 < \cdots < n_r$, *that are realizable are the following*:

(a) $[2, 4, 6, \cdots, 2n]$,

(b) $[4, 8, \cdots, 4n]$,

(c) $[2, 4, 8, \cdots, 4n]$,

(d) $[4, 6, \cdots, 2n]$.

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First of all, it is well known that the types (a), (b), (c), (d) are realizable by $BU(n)$, $BSp(n)$, $CP^\infty \times BSp(n)$ and $BSU(n)$, respectively. Hence, we have to prove that if the type $[n_1, \cdots, n_r]$ is realizable then it coincides with one of the types (a), (b), (c), (d).

The proof is based on the powerful results of [1] and also on the results on algebras over the mod 2 Steenrod algebra, contained in [4] and [5]. For the sake of clarity we list all these results:

(1.1) ([1]) If $A = \mathbf{Z}_p[x_1, \cdots, x_r]$ is a polynomial algebra over the mod p Steenrod algebra and the degrees of x_i, $i = 1, \cdots, r$ are prime to p then A is a product of the irreducible types listed in [3].

(2.1) ([5]) If a polynomial algebra over $\mathbf{Z}_2$ admits an action of the mod 2 Steenrod algebra and has a generator in dimension n then it has a generator in dimension k for all k such that $\binom{k-1}{n-k} \equiv 1$ (2).

To state the results of [4] we need some terminology. Let A be a polynomial algebra over $\mathbf{Z}_2$ and let $[n_1, \cdots, n_r]$, $n_1 < \cdots < n_r$, be the type of A. We say that $[n_1, \cdots, n_r]$ is allowable if the derived truncated polynomial algebra A/D^3 admits the structure of an algebra over the mod 2 Steenrod algebra. We say that $[n_1, \cdots, n_r]$ is irreducible if it is not a disjoint union of two non-empty allowable types. We say that $[n_1, \cdots, n_r]$ is simple if it is allowable, irreducible and contains an odd integer. In [4] the following is proved:

(3.1) (lemma 2.1) If $[n_1, \cdots, n_r]$ is allowable and for some i, t, $n_i < 2^t \leqq n_{i+1}$ and $n \not\in \{n_{i+1}, \cdots, n_r\}$ for $2^t < n < 2^t + 2^{t-1}$, then $[n_1, \cdots, n_i]$ and $[n_{i+1}, \cdots, n_r]$ are both allowable.

(3.2) (lemma 2.3) Suppose $[n_1, \cdots, n_r]$ is allowable. If whenever $n_i = 2^t$ $(i > 1)$ then there exists j so that $2^t < n_j < 2^t + 2^{t+1}$ then $[n_1, \cdots, n_r]$ is irreducible.

(3.3) $[n_1, \cdots, n_r]$ is allowable if and only if $[2n_1, \cdots, 2n_r]$ is.

(3.4) The only simple type containing a single odd entry, with $n_1 = 4$ is $[4, 6, 7]$.

PROPOSITION 2. *If the type $[n_1, \cdots, n_r]$ is realizable then it can be expressed as a union of some of the following types*: (A) $[4, 6, \cdots, 2n]$; (B) $[4, 8, \cdots, 4(n-1), 2n]$; (C) $[4, 8, \cdots, 4n]$; (D) $[4, 12]$; (E) $[4, 24]$; (F) $[2]$; (G) $[12, 16]$; (H) $[4, 12, 16, 24]$, $[4, 10, 12, 16, 18, 24]$, $[4, 12, 16, 20, 24, 28, 36]$, $[4, 16, 24, 28, 36, 40, 48, 60]$.

PROOF. Let us denote by $p_1, \cdots, p_t$ the primes > 7 which divide $n_1, \cdots, n_r$. For each p_i, $i = 1, \cdots, t$, let us choose an integer α_i prime to p_i, such that $\alpha_i \not\equiv \pm 1$

(p_i) and let us consider the following system of congruences:

$x \equiv 3$ (8), (7), (5),

$x \equiv 5$ (9),

$x \equiv \alpha_i$ (p_i).

It is clear that we can find a solution k of this system. Since k is prime to $2, 3, 5, 7, n_i$, $i = 1, \cdots, r$, by a classical theorem of Dirichlet, there is a prime $p > 7$, $p > n_i$, $i = 1, \cdots, r$, such that $p \equiv k$ (N), $N = 2 \cdot 3 \cdot 5 \cdot 7 \cdot n_1 \cdots n_r$. This prime p has the following property: if $n_i = 2m$ and $p \equiv 1$ (m) then $m = 1, 2$ and if $p \equiv -1$ (m) then $m = 1, 2, 3, 4, 6, 12$.

Let X be a space such that $H^*(X; \mathbf{Z}) \cong \mathbf{Z}[x_1, \cdots, x_r]$, $\dim x_i = n_i$. Then $H^*(X; \mathbf{Z}_p) \cong \mathbf{Z}_p[x_1, \cdots, x_r]$ and so the type $[n_1, \cdots, n_r]$ is realizable mod p. Since p does not divide $n_1, \cdots, n_r$, we can apply (1.1): the type $[n_1, \cdots, n_r]$ must be a combination of the irreducible types listed in [3]. But each type in the list in [3] can only occur for some primes p and one easily checks that the choice of p we have made allows only the types (A) to (H) to appear. ■

From (2.1) we can see that not all types (A) to (H) in Proposition 2 are realizable and so the converse of Proposition 2 is certainly not true. If we consider only primes >7 then all types (A) to (H), except (E) and (G), are realizable. However, it is not true that every type which is realizable for all primes >7 is a combination of types (A), (B), (C), (D), (F), (H). For a counterexample, see [1], example 1.4.

PROPOSITION 3. *If $n > 3$ is odd then the types $[4, 8, \cdots, 4(n-1), 2n]$ and $[2, 4, 8, \cdots, 4(n-1), 2n]$ are not realizable.*

PROOF. It clearly suffices to prove the proposition for the second type only. If such an algebra is realizable it should admit an action of the mod 2 Steenrod algebra and so the type $[2, 4, \cdots, 4(n-1), 2n]$ is allowable. By (3.3), $[1, 2, 4, 6, \cdots, 2(n-1), n]$ is also allowable. By (3.1) the type $[4, 6, \cdots, 2(n-1), n]$ is allowable. Hence, by (3.2), $[4, 6, \cdots, 2(n-1), n]$ is simple. Since it contains a single odd entry and $n_1 = 4$, we conclude by (3.4) that $[4, 6, \cdots, 2(n-1), n] = [4, 6, 7]$, a contradiction. ■

PROOF OF THEOREM 1. Let us assume that the type $[n_1, \cdots, n_r]$, $n_1 < \cdots < n_r$, is realizable. From Proposition 2 we get that $[n_1, \cdots, n_r]$ should be a union of the types (A) to (H). Since we are assuming $n_1 < \cdots < n_r$, there is at most one generator in dimension 4.

(2.1) yields that the existence of a generator in dimension 12 implies that there is a generator in dimension 8. Similarly, if there is a generator in dimension 24 then there is also a generator in dimension 16 and, in the same way, a generator in dimension 40 implies a generator in dimension 32. From these facts and Proposition 3 we get that the types (B) ($n > 3$, n odd), (D), (E), (G) and (H) are not realizable.

If there is no generator in dimension 4, we see that the algebra has type (F) and the proof is concluded. Let us assume that there is a generator (and only one) in dimension 4. In this case, $[n_1, \cdots, n_r]$ coincides with one of the types (A), (B), (C), (D), (E) or (H), plus, perhaps, the types [2] and [12, 16]. If [12, 16] does not appear, the proof is concluded because the types (B) (n odd, $n > 3$), (D), (E) and (H) are not realizable. If G = [12, 16] does appear then, since there are no two generators in the same dimension, the possible cases reduce to:

(a) [4, 6, 8, 10, 12, 16], $(\mathrm{A}(n = 5) + \mathrm{G})$;
(b) [4, 6, 8, 12, 16], $(\mathrm{B}(n = 3) + \mathrm{G})$;
(c) [4, 8, 12, 16], $(\mathrm{C}(n = 2) + \mathrm{G})$;
(d) [4, 12, 16, 24], $(\mathrm{E} + \mathrm{G})$;
(e) [4, 12, 16], $(\mathrm{A}(n = 2) + \mathrm{G})$;
(f) [4, 6, 12, 16], $(\mathrm{A}(n = 3) + \mathrm{G})$

(where we omit the possible existence of a generator of degree 2). Case (c) coincides with type $\mathrm{C}(n = 4)$ and so it is realizable. The cases (d), (e) and (f) are not realizable because they contain a generator in dimension 12 but no generator in dimension 8. Thus, it remains only to prove that cases (a) and (b) are not realizable. In order to do this we can for instance consider the prime $p = 23$ and then apply (1.1): the types (a) and (b) should be a combination of the types listed in [3]. Since $23 \not\equiv 1, 3$ (8), the type 12 of [3] cannot appear. All other types from 5 to 37 (see the list of [3]) contain generators in dimensions > 16 and so they cannot appear. If the generator of degree 16 in (a) or (b) comes from type 1 in [3] then there should be a generator in dimension 14 and this is not the case. Since $23 \not\equiv 1$ (8), the generator of degree 16 cannot come from type 3. Hence it can only come from type 2a with $23 \equiv 1$ (m) or from type 2b with $m = 8$. But if $23 \equiv 1$ (m), $m > 1$, then $m = 2, 11, 22$ and since the values $m = 11, 22$ would produce generators of degree > 16, we see that $m = 2$ and so the type which produces the generator of dimension 16 produces also a generator of dimension 4, but no generator of dimension 6. This ends the proof if we notice that the generator of degree 6 cannot come from any type in the list of [3] when $p = 23$, without producing another generator of degree 4. ■

Notice that Theorem 1 is not a consequence of the results of [2] and [4] because the type [4, 6, 8, 12, 16] which we have eliminated using the results of [1] cannot be eliminated using only the results of [2] and [4]. This is clear for [2] and also for [4] if we observe that the algebra of type $[4, 6, 8, 12, 16]$ admits an action of the mod 2 Steenrod algebra because it can be written in the form $2[2, 3] + 2^2[2, 3] + 2^4[1]$.

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